

Fluctuations of the Self-Normalized Sum in the Curie-Weiss Model of SOC

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Abstract

We extend the main theorem of [2] about the fluctuations in the Curie-Weiss model of SOC. We present a short proof using the Hubbard-Stratonovich transformation with the self-normalized sum of the random variables.

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1 Introduction

In [2], Raphaël Cerf and Matthias Gorny designed a Curie-Weiss model of self-organized criticality. It is the model given by an infinite triangular array of real-valued random variables $(X_n^k)_{1 \leq k \leq n}$ such that for all $n \geq 1$, (X_n^1, \dots, X_n^n) has the distribution

$$d\tilde{\mu}_{n,\rho}(x_1, \dots, x_n) = \frac{1}{Z_n} \exp\left(\frac{1}{2} \frac{(x_1 + \dots + x_n)^2}{x_1^2 + \dots + x_n^2}\right) \mathbb{1}_{\{x_1^2 + \dots + x_n^2 > 0\}} \prod_{i=1}^n d\rho(x_i),$$

where ρ is a probability measure on \mathbb{R} which is not the Dirac mass at 0, and where Z_n is the normalization constant. This model is a modification of the generalized Ising Curie-Weiss model by the implementation of an automatic control of the inverse temperature.

For any $n \geq 1$, we denote

$$S_n = X_n^1 + \dots + X_n^n, \quad T_n = (X_n^1)^2 + \dots + (X_n^n)^2.$$

By using Cramér's theory and Laplace's method, Cerf and Gorny proved in [2] that, if ρ satisfies

$$\exists v_0 > 0 \quad \int_{\mathbb{R}} e^{v_0 z^2} d\rho(z) < +\infty \quad (*)$$

and if ρ has a bounded density, then

$$\frac{S_n}{n^{3/4}} \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \left(\frac{4\mu_4}{3\sigma^8} \right)^{1/4} \Gamma\left(\frac{1}{4}\right)^{-1} \exp\left(-\frac{\mu_4}{12\sigma^8}s^4\right) ds.$$

The case where ρ is a centered Gaussian measure has been studied in [6]. This fluctuation result shows that this model is a self-organized model exhibiting critical behaviour. Indeed it has the same behaviour as the critical generalized Ising Curie-Weiss model (see [4]) and, by construction, it does not depend on any external parameter.

This result has been extended in [5] to the case where ρ satisfies some Cramér condition, which is fulfilled in particular when ρ has an absolutely continuous component. However the proof is very technical and it does not deal with the case where ρ is discrete for example.

In this paper we prove that the convergence in distribution of $S_n/n^{3/4}$, under $\tilde{\mu}_{n,\rho}$, is true for any symmetric probability measure ρ on \mathbb{R} which satisfies (*). To this end, we study the fluctuations of the self-normalized sum $S_n/\sqrt{T_n}$. With this term, it is possible to use the so-called Hubbard-Stratonovich transformation as in lemma 3.3 of [4], which is the key ingredient for the proof of the fluctuations theorem in the generalized Ising Curie-Weiss model.

Theorem 1. *Let ρ be a symmetric probability measure on \mathbb{R} which is not the Dirac mass at 0 and which has a finite fifth moment. We denote by σ^2 the variance of ρ and by μ_4 its fourth moment. Then, under $\tilde{\mu}_{n,\rho}$,*

$$\frac{S_n}{n^{1/4}\sqrt{T_n}} \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \left(\frac{4\mu_4}{3\sigma^4} \right)^{1/4} \Gamma\left(\frac{1}{4}\right)^{-1} \exp\left(-\frac{\mu_4}{12\sigma^4}s^4\right) ds.$$

Remark: the hypothesis that ρ has a fifth moment may certainly be weakened by assuming instead that

$$\exists \varepsilon > 0 \quad \int_{\mathbb{R}} |z|^{4+\varepsilon} d\rho(z) < +\infty.$$

We prove theorem 1 in section 2. If we add the hypothesis that ρ satisfies (*) then, under $\tilde{\mu}_{n,\rho}$, T_n/n converges in probability to σ^2 . This result is proved in section 3 of [5] using Cramér's theorem, Varadhan's lemma (see [3]) and a conditioning argument. Moreover

$$\forall n \geq 1 \quad \frac{S_n}{n^{3/4}} = \sqrt{\frac{T_n}{n}} \times \frac{S_n}{n^{1/4}\sqrt{T_n}},$$

and condition (*) implies that ρ has finite moments of all orders. Therefore the following theorem is a consequence of theorem 1 and Slutsky lemma (theorem 3.9 of [1]).

Theorem 2. *Let ρ be a symmetric probability measure on \mathbb{R} which is not the Dirac mass at 0 and such that*

$$\exists v_0 > 0 \quad \int_{\mathbb{R}} e^{v_0 z^2} d\rho(z) < +\infty.$$

Then, under $\tilde{\mu}_{n,\rho}$,

$$\frac{S_n}{n^{3/4}} \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \left(\frac{4\mu_4}{3\sigma^8} \right)^{1/4} \Gamma\left(\frac{1}{4}\right)^{-1} \exp\left(-\frac{\mu_4}{12\sigma^8}s^4\right) ds.$$

2 Proof of theorem 1

Let $(X_n^k)_{1 \leq k \leq n}$ be an infinite triangular array of random variables such that, for any $n \geq 1$, (X_n^1, \dots, X_n^n) has the law $\tilde{\mu}_{n,\rho}$. Let us recall that

$$\forall n \geq 1 \quad S_n = X_n^1 + \dots + X_n^n \quad \text{and} \quad T_n = (X_n^1)^2 + \dots + (X_n^n)^2,$$

and that $T_n > 0$ almost surely. We use the Hubbard-Stratonovich transformation: let W be a random variable with standard normal distribution and which is independent of $(X_n^k)_{1 \leq k \leq n}$. Let $n \geq 1$ and let f be a bounded continuous function on \mathbb{R} . We put

$$E_n = \mathbb{E} \left[f \left(\frac{W}{n^{1/4}} + \frac{S_n}{n^{1/4} \sqrt{T_n}} \right) \right].$$

We introduce $(Y_i)_{i \geq 1}$ a sequence of independent random variables with common distribution ρ . We have

$$\begin{aligned} E_n &= \frac{1}{Z_n \sqrt{2\pi}} \mathbb{E} \left[\int_{\mathbb{R}} f \left(\frac{w}{n^{1/4}} + \frac{Y_1 + \dots + Y_n}{n^{1/4} \sqrt{Y_1^2 + \dots + Y_n^2}} \right) \right. \\ &\quad \left. \times \exp \left(\frac{1}{2} \frac{(Y_1 + \dots + Y_n)^2}{Y_1^2 + \dots + Y_n^2} - \frac{w^2}{2} \right) \mathbb{1}_{\{Y_1^2 + \dots + Y_n^2 > 0\}} dw \right]. \end{aligned}$$

We make the change of variable

$$z = \frac{w}{n^{1/4}} + \frac{Y_1 + \dots + Y_n}{n^{1/4} \sqrt{Y_1^2 + \dots + Y_n^2}}$$

in the integral and we get

$$\begin{aligned} E_n &= \frac{n^{1/4}}{Z_n \sqrt{2\pi}} \mathbb{E} \left[\mathbb{1}_{\{Y_1^2 + \dots + Y_n^2 > 0\}} \right. \\ &\quad \left. \times \int_{\mathbb{R}} f(z) \exp \left(-\frac{\sqrt{n} z^2}{2} + z n^{1/4} \frac{Y_1 + \dots + Y_n}{\sqrt{Y_1^2 + \dots + Y_n^2}} \right) dz \right]. \end{aligned}$$

Let $U_1, \dots, U_n, \varepsilon_1, \dots, \varepsilon_n$ be independent random variables such that the distribution of U_i is ρ and the distribution of ε_i is $(\delta_{-1} + \delta_1)/2$, for any $i \in \{1, \dots, n\}$. Since ρ is symmetric, the random variables $\varepsilon_1 U_1, \dots, \varepsilon_n U_n$ are also independent with common distribution ρ . As a consequence

$$\begin{aligned} E_n &= \frac{n^{1/4}}{Z_n \sqrt{2\pi}} \mathbb{E} \left[\mathbb{1}_{\{U_1^2 + \dots + U_n^2 > 0\}} \right. \\ &\quad \left. \times \int_{\mathbb{R}} f(z) \exp \left(-\frac{\sqrt{n} z^2}{2} + \sum_{i=1}^n \frac{z n^{1/4} \varepsilon_i U_i}{\sqrt{U_1^2 + \dots + U_n^2}} \right) dz \right]. \end{aligned}$$

For any $i \in \{1, \dots, n\}$, we denote (in the case where $U_1^2 + \dots + U_n^2 > 0$)

$$A_{i,n} = \frac{U_i}{\sqrt{U_1^2 + \dots + U_n^2}}.$$

By using Fubini's theorem and the independence of ε_i, U_i , $i \in \{1, \dots, n\}$, we obtain

$$\begin{aligned} E_n &= \frac{n^{1/4}}{Z_n \sqrt{2\pi}} \mathbb{E} \left[\mathbb{1}_{\{U_1^2 + \dots + U_n^2 > 0\}} \int_{\mathbb{R}} f(z) \exp \left(-\frac{\sqrt{n} z^2}{2} \right) \right. \\ &\quad \left. \times \mathbb{E} \left(\prod_{i=1}^n \exp \left(z n^{1/4} \varepsilon_i A_{i,n} \right) \middle| (U_1, \dots, U_n) \right) dz \right]. \\ &= \frac{n^{1/4}}{Z_n \sqrt{2\pi}} \mathbb{E} \left[\mathbb{1}_{\{U_1^2 + \dots + U_n^2 > 0\}} \int_{\mathbb{R}} f(z) \exp \left(-\frac{\sqrt{n} z^2}{2} \right) \right. \\ &\quad \left. \times \exp \left(\sum_{i=1}^n \ln \cosh (z n^{1/4} A_{i,n}) \right) dz \right]. \end{aligned}$$

We define the function g by

$$\forall y \in \mathbb{R} \quad g(y) = \ln \cosh y - \frac{y^2}{2}.$$

It is easy to see that $g(y) < 0$ for $y > 0$. We notice that $A_{1,n}^2 + \dots + A_{n,n}^2 = 1$, so that

$$E_n = \frac{n^{1/4}}{Z_n \sqrt{2\pi}} \mathbb{E} \left[\mathbb{1}_{\{U_1^2 + \dots + U_n^2 > 0\}} \int_{\mathbb{R}} f(z) \exp \left(\sum_{i=1}^n g(z n^{1/4} A_{i,n}) \right) dz \right].$$

Now we use Laplace's method. Let us examine the convergence of the term in the exponential: for any $i \in \{1, \dots, n\}$, the Taylor-Lagrange formula states that there exists a random variable ξ_i such that

$$g(z n^{1/4} A_{i,n}) = -\frac{(z n^{1/4} A_{i,n})^4}{12} + \frac{(z n^{1/4} A_{i,n})^5}{5!} g^{(5)}(\xi_i).$$

By a simple computation, we see that the function $g^{(5)}$ is bounded over \mathbb{R} . As a consequence

$$\begin{aligned} \sum_{i=1}^n g(z n^{1/4} A_{i,n}) &= -\frac{z^4}{12} \frac{(Y_1^4 + \dots + Y_n^4)/n}{((Y_1^2 + \dots + Y_n^2)/n)^2} \\ &\quad + z^5 \frac{(Y_1^5 + \dots + Y_n^5)/n}{((Y_1^2 + \dots + Y_n^2)/n)^{5/2}} O\left(\frac{1}{n^{1/4}}\right). \end{aligned}$$

By hypothesis, the distribution ρ has a finite fifth moment. Hence the law of large numbers implies that

$$\forall z \in \mathbb{R} \quad \sum_{i=1}^n g(z n^{1/4} A_{i,n}) \xrightarrow[n \rightarrow +\infty]{} -\frac{\mu_4 z^4}{12\sigma^4} \quad \text{a.s.}$$

Lemma 3. *There exists $c > 0$ such that*

$$\forall z \in \mathbb{R} \quad \forall n \geq 1 \quad \sum_{i=1}^n g(z n^{1/4} A_{i,n}) \leq -\frac{c z^4}{1 + z^2/\sqrt{n}}.$$

Proof. We define h by

$$\forall y \in \mathbb{R} \setminus \{0\} \quad h(y) = \frac{1+y^2}{y^4} g(y).$$

It is a negative continuous function on $\mathbb{R} \setminus \{0\}$. Since $g(y) \sim -y^4/12$ in the neighbourhood of 0, the function h can be extended to a function continuous on \mathbb{R} by putting $h(0) = -1/12$. Next we have

$$\forall y \in \mathbb{R} \setminus \{0\} \quad h(y) = \frac{1+y^2}{y^2} \times \left(\frac{\ln \cosh y}{y^2} - \frac{1}{2} \right),$$

so that $h(y)$ goes to $-1/2$ when $|y|$ goes to $+\infty$. Therefore h is bounded by some constant $-c$ with $c > 0$. Next we easily check that $x \mapsto x^2/(1+x)$ is convex on $[0, +\infty[$ so that, for any $z \in \mathbb{R}$ and $n \geq 1$,

$$\begin{aligned} \sum_{i=1}^n g(zn^{1/4}A_{i,n}) &\leq -nc \frac{1}{n} \sum_{i=1}^n \frac{(zn^{1/4}A_{i,n})^4}{1+(zn^{1/4}A_{i,n})^2} \\ &\leq -nc \frac{\left(\frac{1}{n} \sum_{i=1}^n (zn^{1/4}A_{i,n})^2\right)^2}{1+\frac{1}{n} \sum_{i=1}^n (zn^{1/4}A_{i,n})^2} = -\frac{cz^4}{1+z^2/\sqrt{n}}, \end{aligned}$$

since $A_{1,n}^2 + \dots + A_{n,n}^2 = 1$. □

If $|z| \leq n^{1/4}$ then $1+z^2/\sqrt{n} \leq 2$ and thus, by the previous lemma,

$$\left| \mathbb{1}_{\{U_1^2 + \dots + U_n^2 > 0\}} \mathbb{1}_{|z| \leq n^{1/4}} \exp \left(\sum_{i=1}^n g(zn^{1/4}A_{i,n}) \right) \right| \leq \exp \left(-\frac{cz^4}{2} \right).$$

Since

$$\mathbb{E} \left[\int_{\mathbb{R}} \|f\|_{\infty} \exp \left(-\frac{cz^4}{2} \right) dz \right] < +\infty,$$

the dominated convergence theorem implies that

$$\begin{aligned} \mathbb{E} \left[\mathbb{1}_{\{U_1^2 + \dots + U_n^2 > 0\}} \int_{\mathbb{R}} \mathbb{1}_{|z| \leq n^{1/4}} f(z) \exp \left(\sum_{i=1}^n g(zn^{1/4}A_{i,n}) \right) dz \right] \\ \xrightarrow{n \rightarrow +\infty} \int_{\mathbb{R}} f(z) \exp \left(-\frac{\mu_4 z^4}{12\sigma^4} \right) dz. \end{aligned}$$

If $|z| > n^{1/4}$ then $1+z^2/\sqrt{n} \leq 2z^2/\sqrt{n}$ and thus, by the previous lemma,

$$\left| \mathbb{1}_{\{U_1^2 + \dots + U_n^2 > 0\}} \mathbb{1}_{|z| > n^{1/4}} \exp \left(\sum_{i=1}^n g(zn^{1/4}A_{i,n}) \right) \right| \leq \exp \left(-\frac{c\sqrt{n}z^2}{2} \right).$$

Hence

$$\mathbb{E} \left[\mathbb{1}_{\{U_1^2 + \dots + U_n^2 > 0\}} \int_{\mathbb{R}} \mathbb{1}_{|z| > n^{1/4}} f(z) \exp \left(\sum_{i=1}^n g(zn^{1/4}A_{i,n}) \right) dz \right] \leq \frac{\|f\|_{\infty} \sqrt{2\pi}}{n^{1/4} \sqrt{c}},$$

and thus

$$\mathbb{E} \left[\mathbb{1}_{\{U_1^2 + \dots + U_n^2 > 0\}} \int_{\mathbb{R}} f(z) \exp \left(\sum_{i=1}^n g(z n^{1/4} A_{i,n}) \right) dz \right] \\ \xrightarrow{n \rightarrow +\infty} \int_{\mathbb{R}} f(z) \exp \left(-\frac{\mu_4 z^2}{12\sigma^4} \right) dz.$$

If we take $f = 1$, we get

$$\frac{Z_n \sqrt{2\pi}}{n^{1/4}} \xrightarrow{n \rightarrow +\infty} \int_{\mathbb{R}} \exp \left(-\frac{\mu_4 z^4}{12\sigma^4} \right) dz.$$

We have proved that

$$\frac{W}{n^{1/4}} + \frac{S_n}{n^{1/4} \sqrt{T_n}} \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \left(\int_{\mathbb{R}} \exp \left(-\frac{\mu_4 z^4}{12\sigma^4} \right) dz \right)^{-1} \exp \left(-\frac{\mu_4}{12\sigma^4} s^4 \right) ds.$$

Since $(n^{-1/4}W)_{n \geq 1}$ converges in distribution to 0, Slutsky lemma (theorem 3.9 of [1]) implies that

$$\frac{S_n}{n^{1/4} \sqrt{T_n}} \xrightarrow[n \rightarrow \infty]{\mathcal{L}} \left(\int_{\mathbb{R}} \exp \left(-\frac{\mu_4 z^4}{12\sigma^4} \right) dz \right)^{-1} \exp \left(-\frac{\mu_4}{12\sigma^4} s^4 \right) ds.$$

By an ultimate change of variables we compute that

$$\int_{\mathbb{R}} \exp \left(-\frac{\mu_4 z^4}{12\sigma^4} \right) dz = \left(\frac{3\sigma^4}{4\mu_4} \right)^{1/4} \Gamma \left(\frac{1}{4} \right).$$

This ends the proof of theorem 1.

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References

- [1] Patrick Billingsley. *Convergence of probability measures*. Wiley Series in Probability and Statistics: Probability and Statistics. John Wiley & Sons Inc., second edition, 1999. A Wiley-Interscience Publication.
- [2] Raphaël Cerf and Matthias Gorny. A Curie–Weiss model of self–organized criticality. *The Annals of Probability*, to appear, 2015.
- [3] Amir Dembo and Ofer Zeitouni. *Large deviations techniques and applications*, volume 38 of *Stochastic Modelling and Applied Probability*. Springer-Verlag, 2010.
- [4] Richard S. Ellis and Charles M. Newman. Limit theorems for sums of dependent random variables occurring in statistical mechanics. *Z. Wahrsch. Verw. Gebiete*, 44(2):117–139, 1978.
- [5] Matthias Gorny. The Cramér condition for the Curie–Weiss model of SOC. *Brazilian Journal of Probability and Statistics*, to appear, 2015.
- [6] Matthias Gorny. A Curie–Weiss model of self–organized Criticality: the Gaussian case. *Markov processes and related fields*, 20(3):563–576, 2014.